**Journal of Computer Science of Newports Institute of Communications and Economics**
**Volume 6, Issue-2015, ISSN: 2226-3683**

# Visualization of Dual Feasible Bases in Primal Space, A Classroom Note

HINA ZAHEER[1], SYED INAYATULLAH[2], MUHAMMAD IMTIAZ[3], HAFSA ATHAR JAFREE[4],

[1, 2, 3, 4]Department of Mathematics, University of Karachi, Karachi,
hina.zaheer@ymail.com
inayat@uok.edu.pk
imtiaz@uok.edu.pk
hajafree@uok.edu.pk

## ABSTRACT

*This article is basically written as a study aid for students in their linear programming course which could be used to revise and to enhance their concepts in this course. In this article we have revisited and presented a simplified proof of Karush-Kuhn-Tucker (KKT) conditions to illustrate the association of both primal and dual basic solutions in primal space. Furthermore, taking the aid of KKT conditions an extension of graphical method for solving two dimensional LPs (which is not new method but rarely found in textbook materials) has been discussed, which can incorporate both primal and dual basic feasible solutions of an LP simultaneously in a single graph. A numerical example has also been given to demonstrate the extension. This article will assist in providing students a better insight of the geometry of weak and strong duality theorem for primal and dual pairs of LPs.*

**Keywords:** *Linear programming, Graphical Method, Farkas' Lemma, Karush-Kuhn-Tucker Conditions.*

## 1. Introduction

Many problems in business and economics deal with problems which involve minimization or maximization of specific performance index, usually of an economic nature like profit, subject to a set of linear constraints. For this exercise to qualify as linear programming the performance index should also be linear. These problems are referred to as linear programming problems.

The development of linear programming began shortly after World War II and the algorithms for finding a solution to a system of linear inequalities first studied by Fourier in the 19th century [7]. Later, several mathematicians such as [6] and [17], rediscovered these algorithms [14].Today because of its tremendous impact in various disciplines linear programming is now being taught to students at undergraduate and graduate levels [9].

Although there are several approaches to solve LP problems like simplex method [5, 4,], interior point method [15], ellipsoid method [16], minimum angle method and artificial method [18, 10, 12, 11, 13, 1, 2]. But, for the sake of developing a better understanding among the students, graphical approach is the most preferred method for two dimensional LP problems. Graphical approach has an edge over simplex method for two dimensional problems because of its simplicity and also since it reveals the entire primal geometrical structure of the given LP. Thus it enables the students to analyze the problem and its components more conveniently. In fact the graphical method is the cornerstone of the evolution of simplex method. Nowadays the graphical method has become an essential tool to elaborate the elementary linear programming problems to the students [19].

Graphical method is easy to generate some intuition through two-dimensional illustrations, and much of this intuition is generalized to spaces of higher dimensions. The graphical method is the more famous method as it is convenient to use and easily understood. Working with only a few variables at a time they allow operations managers to compare projected demand to existing capacity. It gives the geometrical interpretation of the linear programming problem, and helps teachers to explain the geometry behind other more sophisticated methods.

The traditional graphical method commonly available in text books [8], [5], and [3] (Dantzig, 1963), identifies the feasible region of an LP in its activity space by plotting half-spaces of constraints. The points inside or on the feasible region are called the feasible solutions. The bases corresponding to the vertices of the feasible region are called primal feasible bases. The basis (bases) corresponding to the vertex (vertices) which optimizes (maximize/minimize) the objective function is (are) called the optimal basis (bases). To identify the dual feasible bases one must plot the activity space of dual form. Unfortunately the hand plot of primal and dual activity spaces is possible only for a two dimensional LP with 2 constraints. So, apart from optimality, for analysis purpose the traditional graphical method could only be used to identify both primal and dual feasible bases of an instance of LP with just 2 variables and 2 constraints.

For an LP with 2 variables and greater than 2 constraints, the graphical method could only help in identifying primal feasible bases and for an LP with 2 constraints and greater than 2 variables, the graphical method could only help in identifying dual feasible bases.

In this article, first, we have presented a simplified proof of KKT conditions so as to establish a better understanding of relationship between primal and dual forms of an LP. Combination of the traditional graphical method with the KKT conditions provides a way to analyze primal and dual feasible bases for all the problems with either 2 variables and as many constraints or 2 constraints and as many variables. Or in other words for the above types of LP problems this article discussed a way to observe both primal and dual feasible bases in a single plot. This graphical approach is very useful but rare used in textbook materials.

## 1. Some Basic Concepts

### 1.1. A Linear Programming Problem

A system of linear inequalities (or constraints) is a collection of one or more linear inequalities (or constraints) involving the same set of variables, say $x_1,\ldots,x_n$. The problem of maximizing (or minimizing) a linear function subject to a finite number of linear constraints is called linear programming.

A general LP problem is,

$$\begin{aligned} &\text{Maximize} \quad \mathbf{c}^T\mathbf{x} \\ &\text{Subject to} \quad \mathbf{Ax} \le \mathbf{b} \end{aligned} \tag{1}$$

Where $\mathbf{x} \in \Re^n$ is the decision variable vector, $\mathbf{A} \in \Re^{m\times n}, \mathbf{b} \in \Re^m$ and $\mathbf{c} \in \Re^n$ .

The set of points $\mathbf{x}$ in $\Re^n$ satisfying a linear equation $\boldsymbol{\alpha}\mathbf{x} = \beta$ form a hyper-plane in $\Re^n$, where $\beta$ is a scalar and the non-zero row vector $\boldsymbol{\alpha}$ in $\Re^n$ is called the gradient to the hyper-plane. Whereas an open half-space (a closed half-space) is a set of all points $\mathbf{x}$ in $\Re^n$ satisfying a linear inequality $\boldsymbol{\alpha}\mathbf{x} < \beta$ or $\boldsymbol{\alpha}\mathbf{x} > \beta$ ($\boldsymbol{\alpha}\mathbf{x} \le \beta$ or $\boldsymbol{\alpha}\mathbf{x} \ge \beta$).Intersection of finitely many half-spaces form a polyhedral set or a polyhedron in $\Re^n$,that is set of all points satisfying the given collection of constraints $\mathbf{Ax} \le \mathbf{b}$ (constituting theregion of satisfaction or feasible region

$X = \{\mathbf{x} : \mathbf{Ax} \leq \mathbf{b}\}$) forms a polyhedron. The points inside or on the boundary of a feasible region are called feasible solutions while the points outside the feasible region are termed as infeasible solutions. Any constraint $\boldsymbol{\alpha}\mathbf{x} \leq \beta$ is said to be a binding constraint at a feasible point $\mathbf{x}_0 \in X$ if $\boldsymbol{\alpha}\mathbf{x}_0 = \beta$, otherwise it is said to be a non-binding constraint at $\mathbf{x}_0$. Any vector $\mathbf{d} \in X$ is called a (recession) direction of $X$ if $\mathbf{Ad} \leq \mathbf{0}$.

A polyhedral cone in $\Re^n$ is the intersection of finitely many half-spaces whose hyper-planes pass through the origin, that is for a given set of inequalities $\mathbf{Ax} \leq \mathbf{0}$ ,the set $\{\mathbf{x} : \mathbf{Ax} \leq \mathbf{0}\}$ forms a polyhedral cone.

The associated dual linear program of system (1) is:

$$\begin{aligned} &\text{Minimize} && \mathbf{b}^T\mathbf{y} \\ &\text{Subject to} && \mathbf{A}^T\mathbf{y} = \mathbf{c} \\ &&& \mathbf{y} \geq \mathbf{0} \end{aligned}$$

where $\mathbf{y} \in \Re^m$ is the decision variable vector.
Let $\mathbf{x}'$ and $\mathbf{y}'$ are any primal-dual pair of solutions. According to the complementary slackness condition,

$$(\mathbf{y}')^T(\mathbf{Ax}' - \mathbf{b}) = 0$$

That means if a dual variable is non-zero, then the corresponding primal constraint must be binding and if a primal constraint is nonbinding, then the corresponding dual variable must be zero.

## 1.2 Activity and Requirement Spaces

We usually solve a linear programming problem $\{\mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ geometrically in activity space, this system may be represented as

$$\mathbf{a}^i\mathbf{x} \leq b_i, \mathbf{x} \geq \mathbf{0} \quad \text{for } i = 1, \cdots, m.$$

Here, the row vector $\mathbf{a}^i \in \Re^n$ is the $i^{\text{th}}$ row of the matrix $\mathbf{A}$ . This representation shows the product of the $i^{\text{th}}$row vector $\mathbf{a}^i$ with the vector **x**is less than or equal to the $i^{\text{th}}$ component of **b**.

Another space to visualize the above problem is requirement space where we can represent the system $\{\mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ as

$$\sum_{j=1}^{n} \mathbf{a}_j x_j \le \mathbf{b},\ x_j \ge 0, \quad \text{for } j = 1,\cdots,n.$$

Here, $\mathbf{a}_j \in \Re^m$ is the $j^{\text{th}}$ column of the matrix $\mathbf{A}$. The set of all possible linear combinations $\sum_{j=1}^{n} \mathbf{a}_j x_j$ provided $x_j \ge 0$, for $j = 1,\cdots,n$ of vectors $\mathbf{a}_1, \mathbf{a}_2,\cdots,\mathbf{a}_n$ is the cone generated by $\mathbf{a}_1, \mathbf{a}_2,\cdots,\mathbf{a}_n$.In this representation the system $\{\mathbf{Ax} \le \mathbf{b}, \mathbf{x} \ge \mathbf{0}\}$ has a solution if and only if vector $\sum_{j=1}^{n} \mathbf{a}_j x_j$ is component-wise less than or equal to the vector **b**for non-negative $x_j$.

### 1.3 Farkas' Lemma

Let $\mathbf{A} \in \Re^{m\times n}$, $\mathbf{b} \in \Re^m$, $\mathbf{c} \in \Re^n$, $\mathbf{x} \in \Re^n$, $\mathbf{y} \in \Re^m$ then the system

$$\mathbf{Ax} \le \mathbf{0} \text{ and } \mathbf{c}^T \mathbf{x} > 0 \tag{2}$$

has a feasible solution if and only if the system has no solution.

$$\mathbf{A}^T \mathbf{y} = \mathbf{c} \text{ and } \mathbf{y} \ge \mathbf{0} \tag{3}$$

#### 1.3.1 Geometric Interpretation of Farkas' Lemma

Geometrically, $\mathbf{Ax} \le \mathbf{0}$ and $\mathbf{c}^\mathbf{T}\mathbf{x} > 0$ implies that the vector **x** should make a right or an obtuse angle with each non-zero row vector of **A** and the angle between **x** and the vector **c** should be acute. If we plot half-spaces $\{\mathbf{x} : \mathbf{a}^i \mathbf{x} \le 0\}, i = 1\cdots m$ where $\mathbf{a}^i$ is a normal pointing outward from the feasible side of $i^{\text{th}}$half-space, the intersection of these $m$ half-spaces generate a conic feasible region $C = \{\mathbf{x} : \mathbf{Ax} \le \mathbf{0}\}$ .Here in this text, without loss of generality, it is assumed that the constraints $\mathbf{a}^3\mathbf{x} \le 0, \mathbf{a}^4\mathbf{x} \le 0 \cdots \mathbf{a}^m\mathbf{x} \le 0$ are redundant constraints, that is the feasible region $C$ is actually formed by $\mathbf{a}^1\mathbf{x} \le 0$ and $\mathbf{a}^2\mathbf{x} \le 0$, or equivalently all of $\mathbf{a}^3, \mathbf{a}^4 \cdots \mathbf{a}^m$ lie inside the cone generated by $\mathbf{a}^1$and $\mathbf{a}^2$. Figure 1 shows the region of satisfaction $C$ = $\{\mathbf{x} : \mathbf{a}^i \mathbf{x} \le 0\}, i = 1\cdots m$ formed by constraints and the cone generated by their normals $\mathbf{a}^1, \mathbf{a}^2 \cdots \mathbf{a}^m$

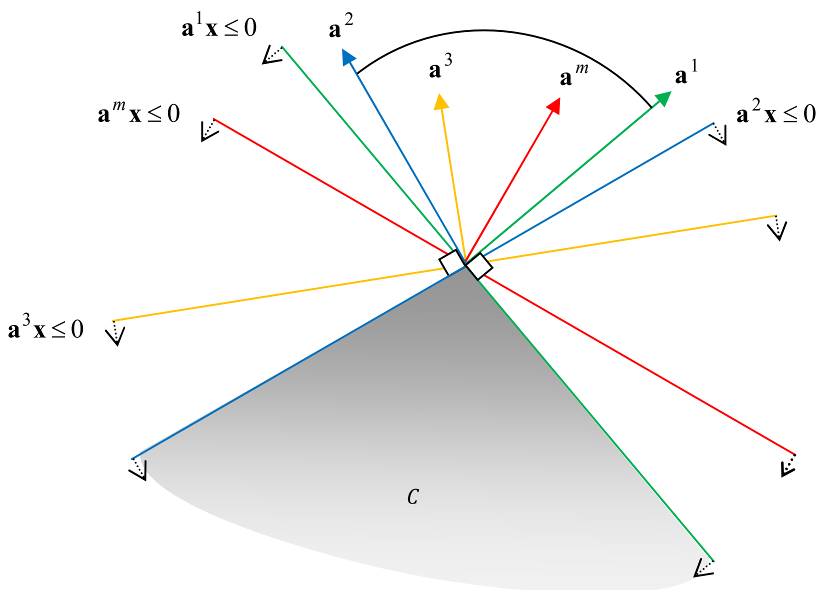


**Figure 1:** Conic feasible region C={x: Ax≤0}

Let $H = \{\mathbf{x} : \mathbf{c}^T\mathbf{x} > 0\}$ be the open half-space with vector **c** pointing towards its feasible side. Thus consistency of system (2) geometrically implies that "the spaces $H$ and $C$ has at least one point common (i.e. $H \cap C \neq \varphi$)". Possible cases of consistency and inconsistency of system (2) are depicted in figure 2 and figure 3 respectively.

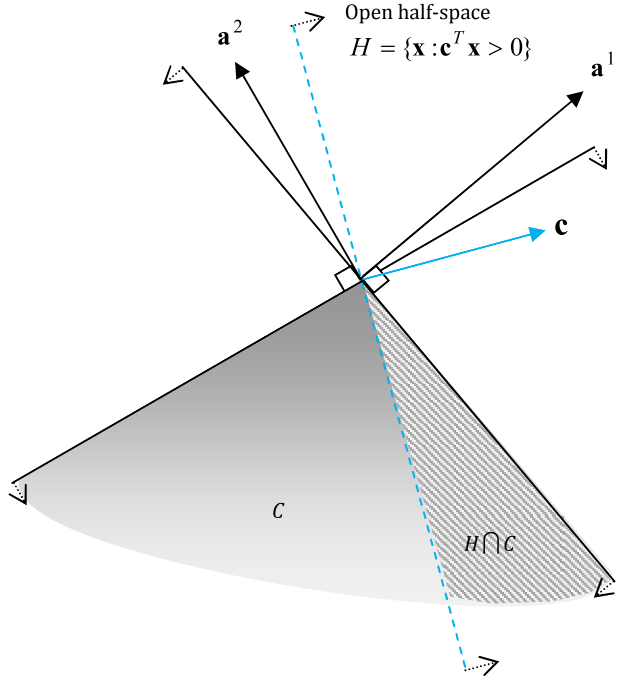


**Figure 2:** System (2) is consistent because H∩C≠ϕ

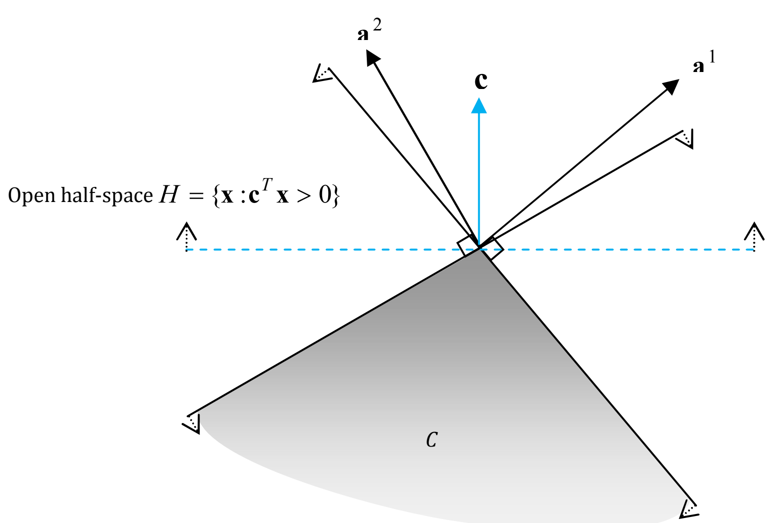


**Figure 3:** System (2) is inconsistent because H∩C≠ϕ

Consider the system (3): $\mathbf{A}^T\mathbf{y} = \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$, Geometrically, in requirement space it means that system (3) has a solution if and only if **c** belongs to the cone generated by the rows of $\mathbf{A}$. Figure 4 shows the case when system (3) does not have a solution as the vector **c** does not lie in the cone generated by the rows of $\mathbf{A}$, but system (2) has a solution since $H \cap C$ is not empty. It is also observable that in this case every **x** in the shaded region has an angle greater than or equal to $90^\circ$ with each $\mathbf{a}^i$ and has an angle less than $90^\circ$ with vector **c**.

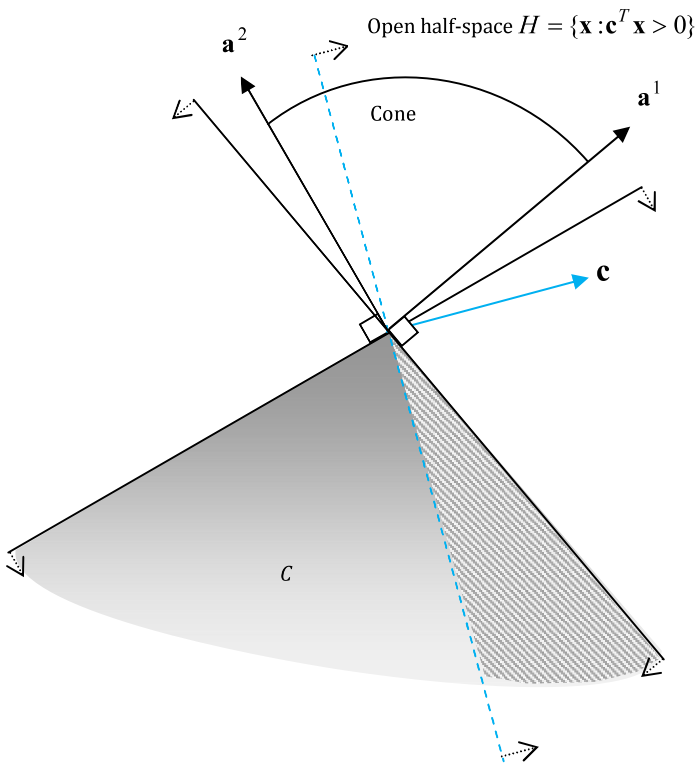


**Figure 4:** System (3) is inconsistent but system (2) is consistent

Figure 5 shows the case when system 3 has a solution since the vector **c** belongs to the cone generated by the rows of $\mathbf{A}$ while system 2 does not have a solution as $H \cap C$ is empty. It is also observable that in this case there is no such **x**, having a right or an obtuse angle with each $\mathbf{a}^i$ and has an acute angle with vector **c**.

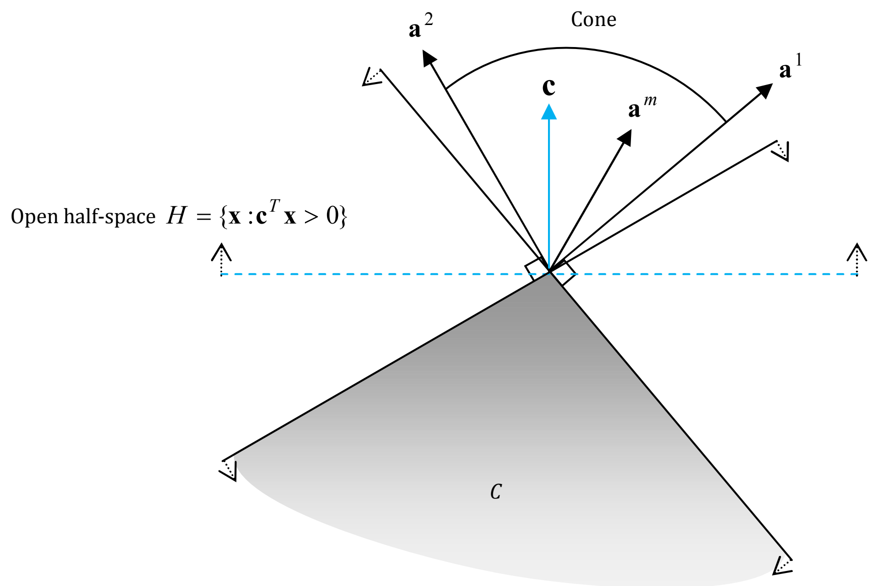


**Figure 5:** System (3) is consistent but system (2) is inconsistent

## 2. Karush-Kuhn-Tucker (KKT) Conditions

Consider the following LP

$$\begin{array}{ll} \text{Maximize} & \mathbf{c}^T\mathbf{x} \\ \text{Subject to} & \mathbf{A}\mathbf{x} \le \mathbf{b} \end{array}$$

where $\mathbf{A} \in \Re^{m\times n}$ and $\mathbf{b} \in \Re^{m}$

Karush-Kuhn-Tucker conditions say that the vector $\mathbf{x}$ is an optimal solution for the above problem if and only if there exists a vector $\mathbf{y} \in \Re^{m}$ such that the following three conditions hold.

$$\mathbf{A}\mathbf{x} \le \mathbf{b} \quad \text{(primal feasibility condition)} \quad (4)$$

$$\mathbf{c} = (\mathbf{A})^T\mathbf{y},\ \mathbf{y} \ge \mathbf{0} \quad \text{(dual feasibility condition)} \quad (5)$$

$$\mathbf{y}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) = \mathbf{0} \quad \text{(complementary slackness condition)} \quad (6)$$

## 2.1 Geometric Interpretation

Condition (4) suggests that the solution $\mathbf{x}$ satisfies the primal in equality which implies that **x** is primal feasible.

Condition (5) means that the vector $\mathbf{c}$ be a non-negative linear combination of rows of $\mathbf{A}$ ,where the coefficients of linear combination are termed as the components of dual vector **y**. Thus condition (5) could be geometrically illustrated as, **c** lies inside the cone generated by row vectors of $\mathbf{A}$ .

Condition (6) means that **x** and **y** both satisfy complimentary slackness condition that is they are primal-dual complementary pair of solutions.

## 2.2 Proof of KKT Conditions

Let $\mathbf{x}^*$ be an optimal basic solution of system (1). We have to show that conditions (4), (5) and (6) hold.

Since $\mathbf{x}^*$ is an optimal solution therefore, it must also be feasible and hence must satisfy condition (4).

Now, let $I=\{i:\mathbf{a}^i\mathbf{x}\le b_i\}$ be the set of indices of binding constraints at $\mathbf{x}^*$. As $\mathbf{x}^*$ is an optimal solution, there does not exist a direction **d** such that $\mathbf{c}^{\mathbf{T}}\mathbf{d}>0$ and $\mathbf{a}^i\mathbf{d}\le 0$, $i\in I$ thus the system $\mathbf{c}^{\mathbf{T}}\mathbf{d}>0$ and $\mathbf{a}^i\mathbf{d}\le 0$, $i\in I$ has no solution. Hence by Farkas' lemma there exists a vector $\mathbf{y}'\in\Re^I$ such that $\sum_{i\in I}(\mathbf{a}^i)^T y_i'=\mathbf{c},\quad y_i'\ge 0,$ for $i\in I$. Let $\mathbf{y}^*=[y_1{}^*,y_2{}^*,..........,y_m{}^*]^T\in\Re^m$ is extended form of the vector $\mathbf{y}'$ such that $y_i{}^*=\begin{cases}y_i' & i\in I\\ 0 & i\notin I\end{cases}$. Hence $\mathbf{y}^*$ satisfies $\sum_{i=1\cdots m}(\mathbf{a}^i)^T y_i{}^*=\mathbf{c},\ y_i{}^*\ge 0,$ which implies satisfaction of condition (5).

As it is described above $y_i{}^*=0$ when $i\notin I$ and $\mathbf{a}^i\mathbf{x}^*-b_i=0$ when $i\in I,$ imply that $y_i{}^*(\mathbf{a}^i\mathbf{x}^*-b_i)=0\ \ ,i=1\cdots m$ which is condition (6).

Conversely, suppose that condition (4), (5) and (6) hold at some point $\mathbf{x}$, we would show that $\mathbf{x}$ must be optimal. Let **y** be complimentary dual point of **x**. According to condition (5)

$$\mathbf{c}^T-\mathbf{y}^T(\mathbf{A})=\mathbf{0}$$

Let $\mathbf{x}'\ne\mathbf{x}$ be any other arbitrary point satisfying condition (4).

$$\Rightarrow(\mathbf{c}^T-\mathbf{y}^T(\mathbf{A}))(\mathbf{x}-\mathbf{x}')=\mathbf{0}(\mathbf{x}-\mathbf{x}')$$

$$\Rightarrow \mathbf{c}^T\mathbf{x} - \mathbf{c}^T\mathbf{x}' - \mathbf{y}^T\mathbf{A}\mathbf{x} + \mathbf{y}^T\mathbf{A}\mathbf{x}' = \mathbf{0}$$

$$\Rightarrow \mathbf{c}^T\mathbf{x} - \mathbf{c}^T\mathbf{x}' - \mathbf{y}^T\mathbf{b} + \mathbf{y}^T\mathbf{A}\mathbf{x}' = \mathbf{0} \qquad \text{From condition (6) } \mathbf{y}^T\mathbf{A}\mathbf{x} = \mathbf{y}^T\mathbf{b}$$

$$\Rightarrow \mathbf{c}^T\mathbf{x} - \mathbf{c}^T\mathbf{x}' - \mathbf{y}^T(\mathbf{b} - \mathbf{A}\mathbf{x}') = \mathbf{0}$$

$$\Rightarrow \mathbf{c}^T\mathbf{x} - \mathbf{c}^T\mathbf{x}' \geq \mathbf{0} \qquad \because \mathbf{b} - \mathbf{A}\mathbf{x}' \geq \mathbf{0} \text{ and } \mathbf{y} \geq \mathbf{0}$$

$$\Rightarrow \mathbf{c}^T\mathbf{x} \geq \mathbf{c}^T\mathbf{x}'$$

Arbitrariness in choice of $\mathbf{x}'$ proves optimality of $\mathbf{x}$.

## 3. Graphical Approach to Identify Primal and Dual Feasible Solutions

**Step 1:** Plot the system of inequalities and identify region of satisfaction. According to condition (4), the corner points on the region of satisfaction are primal feasible (i.e. bases corresponding to these corner points are primal feasible)

**Step 2:** For each corner point sketch a cone is formed by gradients of the binding constraints. Sketch the gradient of objective vector **c** originated at each corner point. According to condition (5), if **c** lies on or inside the cone, the corner point would be dual feasible (i.e. bases corresponding to these corner points are dual feasible).

**Step 3:** The corner points which are both primal and dual feasible (found in step 1 and step 2) are optimal points.

### 3.1. Proof of Correctness

Steps 1, 2 and 3 are clear geometric implication of condition (4), (5) and (6) respectively (see geometric interpretation of KKT conditions defined earlier), which therefore prove the optimality of resulting corner points.

**Example 1**

$$\begin{array}{ll} \text{Maximize} & 3\mathbf{x_1} + 5\mathbf{x_2} \\ \text{Subject to} & 3\mathbf{x_1} + 2\mathbf{x_2} \leq 18 \\ & \mathbf{x_1} - \mathbf{x_2} \leq 3 \\ & -2\mathbf{x_1} - \mathbf{x_2} \leq -2 \\ & 2\mathbf{x_2} \leq 12 \\ & \mathbf{x_1} \leq 4 \\ & -\mathbf{x_1} \leq 0 \\ & -\mathbf{x_2} \leq 0 \end{array} \qquad (7)$$

Here $\mathbf{c} = [3,5]^T$, $\mathbf{a}^1 = [3,2]$, $\mathbf{a}^2 = [1,-1]$, $\mathbf{a}^3 = [-2,-1]$, $\mathbf{a}^4 = [0,2]$, $\mathbf{a}^5 = [1,0]$, $\mathbf{a}^6 = [-1,0]$ and $\mathbf{a}^7 = [0,-1]$.

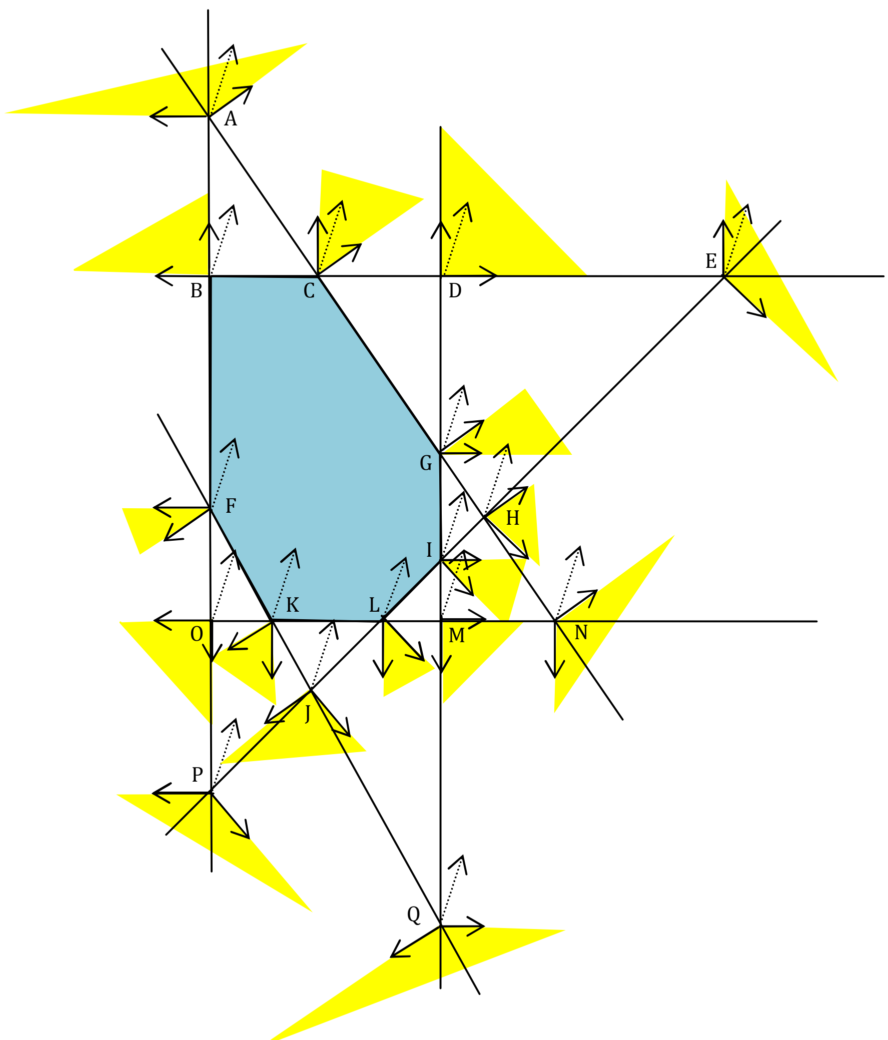


**Figure 6:** Feasible region and cones generated by gradients of binding constraints of system (7)

It is clear from figure 6 that all corner points of the region of satisfaction that is C, G, I, L, K and F are primal feasible. To identify the complimentary dual feasible points we must find the point of intersection at which the gradient of the objective function is a linear combination of the gradients of the binding constraints. Clearly the bases corresponding to the points A, C, D and E are dual feasible as the gradient $\mathbf{c} = [3,5]^T$ lies in the cone generated by the gradients $\mathbf{a}^1$ and $\mathbf{a}^6$, $\mathbf{a}^1$ and

$\mathbf{a}^4$, $\mathbf{a}^4$ and $\mathbf{a}^5$, $\mathbf{a}^2$ and $\mathbf{a}^4$. The point *C* is optimal solution of system (7) since it is both primal and dual feasible (see table 1).

**Table 1:** Corner points and their feasibility status of system (7)

| **Corner Points** | **Status** | |
|---|---|---|
| A | Primal Infeasible | Dual Feasible |
| B | Primal Feasible | Dual Infeasible |
| C | Primal Feasible | Dual Feasible |
| D | Primal Infeasible | Dual Feasible |
| E | Primal Infeasible | Dual Feasible |
| F | Primal Feasible | Dual Infeasible |
| G | Primal Feasible | Dual Infeasible |
| H | Primal Infeasible | Dual Infeasible |
| I | Primal Feasible | Dual Infeasible |
| O | Primal Infeasible | Dual Infeasible |
| K | Primal Feasible | Dual Infeasible |
| L | Primal Feasible | Dual Infeasible |
| M | Primal Infeasible | Dual Infeasible |
| N | Primal Infeasible | Dual Infeasible |
| J | Primal Infeasible | Dual Infeasible |
| P | Primal Infeasible | Dual Infeasible |
| Q | Primal Infeasible | Dual Infeasible |

Primal and dual basic solutions can be obtained by using simple algebraic computations. Table 2 displays the primal and dual basic solutions for each corner point.

**Table 2:** Corner points and their corresponding primal and dual basic solutions of system (7)

| **Corner Points** | **Primal basic solutions** | **Dual basic solutions** |
|---|---|---|
| A | (0,9,0,12,7,-6,4) | (9/2,0,5/2,0,0,0,0) |
| B | (0,6,6,9,4,0,4) | (-3,0,0,0,0,5/2,0) |
| C | (2,6,0,7,8,0,2) | (0,0,1,0,0,3/2,0) |
| D | (4,6,-6,5,12,0,0) | (0,0,0,0,0,5/2,3) |
| E | (9,6,-21,0,22,0,-5) | (0,0,0,3,0,5/2,0) |
| F | (0,2,14,5,0,8,4) | (-3,0,0,0,-5,0,0) |
| G | (4,3,0,2,7,6,0) | (0,0,5/2,0,0,0,-9/2) |
| H | (24/5,9/5,0,0,47/5,42/5,-4/5) | (0,0,8/5,-9/5,0,0,0) |
| I | (4,1,4,0,7,10,0) | (0,0,0,-5,0,0,8) |
| O | (0,0,18,3,-2,12,4) | (-3,-5,0,0,0,0,0) |
| K | (1,0,15,2,0,12,3) | (0,-7/2,0,0,-3/2,0,0) |
| L | (3,0,9,0,4,12,1) | (0,-8,0,3,0,0,0) |
| M | (4,0,6,-1,6,12,0) | (0,-5,0,0,0,0,3) |
| N | (6,0,0,-3,10,12,-2) | (0,-3,1,0,0,0,0) |
| J | (5/4,-4/3,47/3,0,0,44/3,7/3) | (0,0,0,-7/3,-8/3,0,0) |
| P | (0,-3,24,0,-5,18,4) | (-8,0,0,-5,0,0,0) |
| Q | (4,-6,18,-7,0,24,0) | (0,0,0,0,-5,0,-7) |

**Conclusion:**

Basically, this article is written to encourage students and teachers to use an extended graphical method which can be used to solve both primal and dual forms of an LPP simultaneously. The method is not new but without any reason rarely found in textbook materials. This approach is advantageous as it provides primal feasible solutions (if exists) and as well as the dual basic feasible solutions (if exists) in primal space. Finally, the numerical example has also been given to demonstrate the difference between the traditional graphical method and the proposed approach.

## References

[1] Arsham, H. (2006). A Big-M Free Solution Algorithm for General Linear Programs. International Journal of Pure and Applied Mthematics, 32.

[2] Arsham, H. (1997b). An Artificial-Free Simplex Algorithm for General LP Models,. Mathematical and Computer Modelling, 25 (1), 107-123.

[3] Bazaraa, M. S., Jarvis, J. J., & Sherali, H. D. (1977). Linear Programming and Network Flows.

[4] Bland, R. (1977). New finite pivoting rules for the simplex method. Mathematical Operation Research, 2, 103-107.

[5] Dantzig, G. (1963). Linear programming and Extensions. Princeton Univ. Press, NJ.

[6] Dines, L. (1918). Systems of inequalities. Annal of Mathematics, 20 (2), 191-198.

[7] Fourier, J. B. (n.d.). Second extrait . (G. Darboux, Ed.) Oeuvres , 325-328.

[8] Hillier, F. S., & Lieberman, G. J. (2003). Introduction to Operations Research (7th ed.). New Delhi: Tata Mcgraw-Hill Publishing Company Limited.

[9] Hosein, A., Aczel, J., & Clow, D. (2006), "The Teaching of Linear Programming in Different Disciplines and in Different Countries", in 3rd International Conference on the teaching of Mathematics at Undergraduate Levels. , 1.

[10] Inayatullah, S. (2012), "Some New Artificial-Free Pivotal Algorithms for Linear Programming". Ph.d thesis, Faculty of science, Department of Mathematical Sciences, University of Karachi.

[11] 1 Inayatullah, S., Khan, N., & Imtiaz, M. (2009), "New artificial-free Phase Simplex Method", International Journal of Basic & Applied Sciences IJBAS, 9 (10), 97- 114.

[12] Inayatullah, S., Khan, N., Imtiaz, M., & Khan, F. H. (2010), "A Minimum Angle Method for Dual Feasibility", Arch. Appl. Sci. Res. , 28-36.

[13] Inayatullah, S., Khan, N., Imtiaz, M., & Khan, F. (2010), "New Minimum Angle Algorithms For Feasibility And Optimality", Canadian Journal on Computing in Mathematics, Natural Sciences, Engineering & Medicines, 1 (1), 22-36.

[14] Kaluzny, B. (2001), "Finite Pivot algorithms and Feasibility", MS thesis, Faculty of Graduate Studies and Research, School of Computer Science, McGill University, Montreal, Quebec, Canada.

[15] Karmarkar, N. (1984), "A New Polynomial Time Algorithm for Linear Programming", Combinatorica, 4, 373-395.

[16] Khachian, L. (1980), "Polynomial algorithms in linear programming. Computational Mathematics and Mathematical Physics", 20, 53-72.

[17] Motzkin, T. (1952), "Contributions to the theory of linear inequalities", RAND Corporation Translation 22 .

[18] Pan, P.-Q. (1997), "The most-obtuse-angle row pivot rule for achieving dual feasibility: a computational study", European journal of operational research, 101, 167-176.

[19] Pereira, J. C., & Fernandes, S. (2013), "Two-Variable Linear Programming: A Graphical Tool With Marhematica. Symcomp".